The Mathematical Association of America

The Edge of the Universe

Celebrating Ten Years of Math Horizons

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Fermat Faces Reality

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A Diophantine Drama in One Act

Student with new calculator says to Teacher out of the blue: "Is it really true that $27^2 + 18^3 = 9^4$?

Teacher to Student: "Why, ahem! Yes. [Ducking under the desk to check.] How'd you notice that? Has a nice symmetry to it, doesn't it? Relies on $1 + 2^3 = 3^2$, probably a coincidence of sorts."

Teacher, having burned the midnight oil, goes right on the Next Day: "It was. But, you know, this power equation you mentioned works with other numbers. For instance: $648^2 + 108^3 = 36^4$. Try factoring out a few 6's and using a little algebra to check this one, instead of just multiplying things out and adding. In fact, the only addition you should need to perform is $1 + 3 = 4$."

Student, having burned the midnight oil, continues on the Third Day: "Hey! A little factoring and exponents and you were right about 648^2 plus 108^3 equaling 36^4. It just says 1 plus 3 equals 4, all multiplied by 419,904—which is 9 times 6 to the 6th power. This is cool. Tell me more."

Teacher, getting wiser: "Well, if you really want to be convinced algebra is useful, try checking a larger one like 110,592 squared plus 4608 cubed equals 576 to the fourth. See what use your calculator is to you on that. But, you can fiddle with those big numbers at home tonight. Right now, be amazed. Take two minutes to check that $28^2 + 8^3 = 6^4$ too, but for a different reason."

Student "Fantasmic! But whaddaya mean, different reason?"

Teacher mumbles off-handedly: "Oh, not like $1 + 3 = 4$ or $1 + 2^3 = 3^2$ more like $7^2 + 2^5 = 3^4$."

Student says: "Huh?"

Illustration by Gregory Nemec

Teacher continues unabated: "After you've checked the triple (28, 8, 6), use a little algebra and your calculator (you shouldn't really need it, though) to figure out what $1{,}176^2 + 49^3$ is the fourth power of—excuse me, of what this sum is the fourth power. You'll discover that you were wrong in thinking that the Pythagorean triple (7, 24, 25) had no practical applications."

Student says: "Huh? I thought what?"

Teacher fires the supposed zinger, even further over the student's head: "But, if you want a really interesting one for tonight, check that $433^2 + 143^3 = 42^4$. You probably won't get much help from exponents on this one, since 433 is prime. For you, it may be a calculator job all the way. And it works for mysterious reasons known only to the Great Number Wielder in the Sky."

Student "Hey, Teach! How do you know all of this stuff? I mean, does it just come to you?"

Teacher "Vast experience and years of mathematical study, O inquisitive one—experience and study. Not to mention a large dose of the mathematician's secret weapon, C & P."

Student "What's C & P?"

Teacher "Calculation and Perseverance. It's usually applied with HuTSPA, which is the maxim that keeps us looking smart: **H**ide **u**re **T**ons of **S**cratch **P**aper **A**fterward."

Student "Yeah, yeah. A real knee-slapper. Listen I gotta go, but could I ask just one more question? This guy Fermat: After he and his buddies noticed that these Pythagorean triples you told us about were so easy and then showed that $x^3 + y^3 = z^3$ was a loser, why didn't he next take up $x^2 + y^3 = z^4$, like we're doing, instead of going to all fourth powers — another loser?"

Teacher "He lacked imagination, I guess."

Student, missing the joke: "Yeah. His equation was sorta dumb. Boring. No solutions at all. Ours is good for something — for us. We've already found five solutions (x, y, z) with y less than 150."

Teacher "Yup, 'we' have. There are six more with $y \leq 150$ that you should try to find. One that has $y = 128$ is really not new—it's (28, 8, 6) in disguise. Of the five additional "primitive" solutions, one has y prime, and in another z is prime—types not easy to come by. But 11 is all there are.*" ■

*Pssst! Ecoutez moi! The Teach has found a truly unremarkable proof of this. Fortunately, the space down here is much too small to contain it. Vive le Professeur Wiles! PdF.